Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 5, pp. 781–788, 2008

PROBLEM OF FILLING OF A SPHERICAL CAVITY IN A KELVIN–VOIGT MEDIUM

S. V. Osipov

UDC 532.135

This paper is concerned with mathematical modeling and solution of the problem of the collapse of a spherical cavity in a viscoelastic medium under the action of constant pressure at infinity. A differential equation of motion for the cavity boundary is constructed and solved numerically. The existence of three modes of motion of the boundary is established, and a map of these modes in the plane of the determining parameters is constructed. Asymptotic forms of the solutions of the problem for all modes are constructed. The problem of cavity collapse with capillary forces taken into account is formulated and solved.

Key words: *Kelvin–Voigt medium, viscoelastic medium, cavity filling, finite strains.*

Problems of bubble collapse in fluids have been studied by various researchers. Rayleigh was one of the first to examine the problem of filling of a spherical cavity in an inviscid incompressible fluid. He established [1] that, at the end of the filling process, the velocity of the cavity surface directed to the center increases without bound as $r^{-3/2}$, i.e., unlimited energy cumulation occurs. This can cause rapid wear of propellers and turbines operating under cavitation conditions: bubble collapse on a metal surface can lead to rapid fracture of this surface. The Rayleigh problem for a viscous fluid is studied in [2, 3] (the results of these studies are also given in [4]). It is shown that there are two modes of cavity filling, depending on the initial cavity radius: at a radius smaller than the critical one, filling occurs in infinite time, and energy cumulation is completely eliminated by viscosity; for a large initial radius, the cavity collapses rapidly with unlimited energy cumulation in the stage of focusing. An elegant solution of the problem of cavity filling in a viscous fluid was obtained, independently of other researchers, by Zababakhin [5]; therefore, this problem is often called the Zababakhin problem.

Andreev and Gal'perin, studying the behavior of bubbles in a viscous fluid with capillary forces taken into account, obtained a different result: if the initial bubble radius is small (smaller than the critical value), the rate of filling decreases, but not to zero, and the bubble is filled in finite time [6].

Diffusion dissolution of bubbles in a relaxing medium was examined by Zana and Leal [7, 8] and diffusion growth by Yoo and Han [9, 10]. Numerical analysis of the collapse of an empty cavity and nonlinear oscillations of bubbles in Oldroyd, Rivlin–Ericksen, and Jeffrery fluids and in a fluid with a strain rate lag was performed by Pearson and Middleman [11] and Ting [12].

It is of interest to study the motion of the bubble boundary in the presence of elastic forces. In the present paper, we consider the problem of filling of a spherical cavity in a viscoelastic Kelvin–Voigt medium under the action of constant pressure at infinity.

1. The Kelvin–Voigt medium is given by the equation of state

$$P = -pI + 2\mu D + 2\rho\varkappa E, \tag{1}$$

where P is the stress tensor, D is the strain rate tensor, E is the finite strain tensor, p is the pressure, μ, ρ, and $\varkappa$ are the viscosity, density, and specific shear modulus, respectively, which are constants. Because the symmetry

Tyumen' Petroleum Research Center, Tyumen 625000; svosipov@tnk-bp.com. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 5, pp. 93–101, September–October, 2008. Original article submitted July 24, 2007; revision submitted January 28, 2008.

of motion is spherical, the tensor P is diagonal and $P_{\theta\theta} = P_{\varphi\varphi}$. The tensor D is also diagonal, $D_{rr} = \partial u/\partial r$, and $D_{\theta\theta} = D_{\varphi\varphi} = u/r$.

To determine the components of the strain tensor E in spherical coordinates, we use the representation

$$x_1 = r\cos\varphi\sin\theta, \qquad x_2 = r\sin\varphi\sin\theta, \qquad x_3 = r\cos\theta,$$

$$X_1 = \xi\cos\varphi\sin\theta, \qquad X_2 = \xi\sin\varphi\sin\theta, \qquad X_3 = \xi\cos\theta.$$

Here X_i, $\xi = (r^3 - s(t)^3 + a^3)^{1/3}$ and x_i and r are the Lagrangian and Eulerian coordinates in Cartesian and spherical systems, respectively, $r = s(t)$ is the free boundary of the bubble, and a is the initial radius. The finite strain tensor is given by the relation (see [13, 14])

$$E_{ij} = \frac{1}{2}\Big(I_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\Big).$$

Since

$$X_i = x_i\frac{((x_1^2 + x_2^2 + x_3^2)^{3/2} - s^3 + a^3)^{1/3}}{(x_1^2 + x_2^2 + x_3^2)^{1/2}},$$

we have

$$\frac{\partial X_i}{\partial x_j} = \delta_{ij}\,\frac{((x_1^2 + x_2^2 + x_3^2)^{3/2} - s^3 + a^3)^{1/3}}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}$$

$$+\, x_i x_j\,\frac{s^3 - a^3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}((x_1^2 + x_2^2 + x_3^2)^{3/2} - s^3 + a^3)^{2/3}}$$

$$= \delta_{ij}\,\frac{(r^3 - s^3 + a^3)^{1/3}}{r} + x_i x_j(r,\varphi,\theta)\frac{s^3 - a^3}{r^3(r^3 - s^3 + a^3)^{2/3}}$$

(δ_{ij} is the Kronecker symbol).

Because the problem has spherical symmetry, we assume that $\cos\varphi\sin\theta = 1$. As a result, in the spherical coordinates, we have

$$2E_{rr} = 1 - r^4(r^3 - s^3 + a^3)^{-4/3}, \qquad 2E_{\theta\theta} = 2E_{\varphi\varphi} = 1 - r^2(r^3 - s^3 + a^3)^{2/3}.$$

Using the formula for calculating the divergence tensor in spherical coordinates and taking into account the symmetry of motion, from the general momentum equation $\rho\,d\boldsymbol{v}/dt = \operatorname{div} P$, we obtain

$$\rho\Big(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r}\Big) = \frac{1}{r^2}\,\frac{\partial\,(r^2 P_{rr})}{\partial r} - \frac{2P_{\theta\theta}}{r}. \tag{2}$$

Denoting $P_{rr} = Q$ and introducing a new unknown function $P_{rr} - P_{\theta\theta} = R$, we write, Eq. (2) as

$$\rho\Big(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial r}\Big) = \frac{\partial Q}{\partial r} + \frac{2R}{r}. \tag{3}$$

Because the Kelvin–Voigt fluid is incompressible, the continuity equation for this medium in spherical coordinates is written as

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = 0. \tag{4}$$

The rheological relation (1) for a Kelvin–Voigt medium formally contains three equations corresponding to spatial coordinates, but the second and third equations coincide; therefore, two independent equations remain:

$$Q = -p + 2\mu\,\frac{\partial u}{\partial r} + \rho\varkappa[1 - r^4(r^3 - s^3 + a^3)^{-4/3}],$$

$$Q - R = -p + 2\mu u/r + \rho\varkappa[1 - r^{-2}(r^3 - s^3 + a^3)^{2/3}].$$

The pressure can be eliminate from these equations (since the dynamic condition on the free boundary contains only the function Q). As a result, using equality (4) and taking into account the range of definition, we obtain

$$R = -6\mu u/r + \rho\varkappa[r^{-2}(r^3 - s^3 + a^3)^{2/3} - r^4(r^3 - s^3 + a^3)^{-4/3}], \qquad t > 0, \quad r > s(t). \tag{5}$$

System (3)–(5) is closed, and Eq. (4) is integrable:

$$u = s^2\dot{s}/r^2, \qquad t > 0, \quad r \geqslant s(t). \tag{6}$$

Here the kinematic condition $u = \dot{s}$ on the free boundary of the bubble $r = s(t)$ is taken into account; the dot denotes the derivative with respect to t. Substitution of expression (6) into Eqs. (3) and (5) yields

$$\frac{\partial Q}{\partial r} + \frac{2R}{r} = \rho\Big(\frac{s^2\ddot{s} + 2s\dot{s}^2}{r^2} - \frac{2s^4\dot{s}^2}{r^5}\Big), \qquad t > 0, \quad r > s(t); \tag{7}$$

$$R = -6\mu s^2\dot{s}/r^3 + \rho\varkappa[r^{-2}(r^3 - s^3 + a^3)^{2/3} - r^4(r^3 - s^3 + a^3)^{-4/3}], \qquad t > 0, \quad r > s(t). \tag{8}$$

Eliminating the function R from system (7), (8), we obtain the equation

$$\frac{\partial Q}{\partial r} = 12\mu\,\frac{s^2\dot{s}}{r^4} - 2\rho\varkappa[r^{-3}(r^3 - s^3 + a^3)^{2/3} - r^3(r^3 - s^3 + a^3)^{-4/3}]$$

$$+ \rho\Big(\frac{s^2\ddot{s} + 2s\dot{s}^2}{r^2} - \frac{2s^4\dot{s}^2}{r^5}\Big), \qquad t > 0, \quad r > s(t). \tag{9}$$

The initial condition for (9) is written as

$$Q = 0, \qquad t > 0, \qquad r = s(t). \tag{10}$$

Integration of Eq. (9) with the use of condition (10) yields

$$Q + c = -4\mu\,\frac{s^2\dot{s}}{r^3} - \rho\varkappa\,\frac{r^3 + s^3 - a^3}{r^2(r^3 - s^3 + a^3)^{1/3}} + \rho\Big(-\frac{s^2\ddot{s} + 2s\dot{s}^2}{r} + \frac{s^4\dot{s}^2}{2r^4}\Big), \tag{11}$$

where

$$c = -4\mu\,\frac{\dot{s}}{s} - \rho\varkappa\frac{2s^3 - a^3}{s^2 a} - \rho\Big(s\ddot{s} + \frac{3}{2}\,\dot{s}^2\Big).$$

Letting r tend to infinity and using the limiting value $Q = -p_0$ ad $r \to \infty$, from equality (11) we obtain a nonlinear differential equation of the second order for the function $s(t)$ with known initial conditions:

$$\ddot{s} + \frac{3}{2}\,\frac{\dot{s}^2}{s} + 4\nu\,\frac{\dot{s}}{s^2} + \varkappa\,\frac{2s^3 - a^3}{s^3 a} + \Big(\frac{p_0}{\rho} - \varkappa\Big)\frac{1}{s} = 0; \tag{12}$$

$$s(t) = a, \qquad \dot{s}(t) = 0, \qquad t = 0. \tag{13}$$

The above equation is solved numerically. For convenience, we make the variables nondimensional, introducing the quantities a/Re and $\sqrt{p_0/\rho}$ as the length and velocity scales, respectively [$\operatorname{Re} = (a/\nu)\sqrt{p_0/\rho}$ is the Reynolds number]. As a result, Eq. (12) with initial conditions (13) becomes

$$\ddot{s} + \frac{3}{2}\,\frac{\dot{s}^2}{s} + 4\,\frac{\dot{s}}{s^2} + (1 - M)\,\frac{1}{s} - M\operatorname{Re}^2\,\frac{1}{s^3} + 2\,\frac{M}{\operatorname{Re}} = 0; \tag{14}$$

$$s(t) = \operatorname{Re}, \qquad \dot{s}(t) = 0, \qquad t = 0, \tag{15}$$

where $M = \varkappa\rho/p_0$.

From the solution of problem (14), (15), it follows that several modes of motion of the cavity boundary exists (Fig. 1 gives a map of these modes in the plane (M, Re)]:

1) collapse (region I);

2) monotonic decrease in the radius to a certain value $s > 0$ (region II);

3) nonmotonic decrease in the radius to a certain value $s > 0$ (region III).

In the first case, the rate of collapse increases to infinity and in the second and third case, it is equal to zero.

Figures 2–4 show solutions $s(t)$ of problem (14), (15) for various values of the parameters Re and M (curves 1).

For mode 1, in which the cavity collapses, it is possible to construct the asymptotics of the solution of Eq. (14) at zero (at the moment of collapse). For this, we isolate the main terms, using the representation $s(t) = A(t_* - t)^n$. As a result, we obtain the following equation for the coefficients A and n:

$$\Big(\frac{5n^2}{2} - n\Big)A(t_* - t)^{n-2} - \frac{4n}{A}\,(t_* - t)^{-n-1} - \frac{M\operatorname{Re}^2}{A^3}\,(t_* - t)^{-3n} = 0.$$

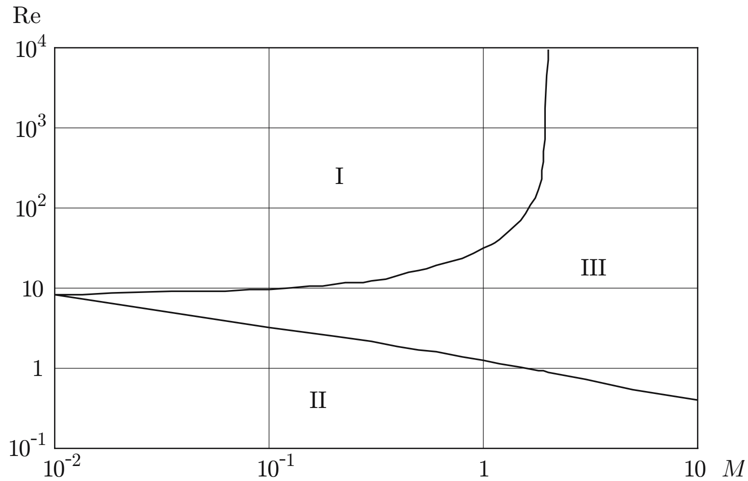

Fig. 1. Modes of behavior of the cavity boundary in the plane (M, Re).

We find the exponent $n = 1/2$. In this case, we obtain the biquadratic equation

$$A^4 - 16A^2 - 8M\,\mathrm{Re}^2 = 0.$$

Since $s(t) \geqslant 0$, this equation has two roots

$$A_{1,2} = \sqrt{8 \pm (64 + 8M\,\mathrm{Re}^2)^{1/2}}\,,$$

of which only the root with a non-negative discriminant has a physical meaning. As a result, the asymptotic behavior of the boundary at the moment of cavity collapse can be described by the expression

$$s(t) = [8 + (64 + 8M\,\mathrm{Re}^2)^{1/2}]^{1/2}(t_* - t)^{1/2}, \qquad t \to t_*.$$

Accordingly, the expression for the rate of collapse is written as

$$\dot{s}(t) = -[8 + (64 + 8M\,\mathrm{Re}^2)^{1/2}]^{1/2}(t_* - t)^{-1/2}/2, \qquad t \to t_*$$

or

$$u(s) = -[8 + (64 + 8M\,\mathrm{Re}^2)^{1/2}]/(2s), \qquad s \to 0.$$

Passing to the elasticity limit as $M \to 0$, we have $u(s) = -8/s$ and $s \to 0$, which agrees with the solution of the Zababakhin problem of cavity collapse in a viscous fluid [5].

For modes 2 and 3, where the elastic forces are great enough to prevent the cavity from collapsing, the limiting radius of the cavity boundary d can be determined from Eq. (14). Taking into account that the velocity and acceleration of motion of the boundary tend to zero [$s(t) \to d$, $\dot{s}(t) \to 0$, $\ddot{s}(t) \to 0$], we obtain the equation for the constant d:

$$2Md^3 + \mathrm{Re}\,(1 - M)d^2 - M\,\mathrm{Re}^3 = 0; \tag{16}$$

the real root of this equation is the limiting value of d.

Using the notation $u = \dot{s}$ and $r = s(t)$, we reduce the order of Eq. (14):

$$\frac{du}{ds} + \frac{3}{2}\,\frac{u}{s} + \frac{4}{s^2} + (1 - M)\,\frac{1}{su} - M\,\mathrm{Re}^2\,\frac{1}{s^3 u} + 2\,\frac{M}{\mathrm{Re}\,u} = 0. \tag{17}$$

Let us examine Eq. (17) near the singular point $s = d$, $u = 0$. For this, we make the substitution $q = s - d$ and, using relation (16), write (17) as

$$\frac{du}{dq} = -\frac{(4\,\mathrm{Re}\,(1 - M)d + 12Md^2)q + (3\,\mathrm{Re}\,d^2 + 8\,\mathrm{Re}\,d)u + \varphi(q, u)}{2\,\mathrm{Re}\,d^3 u + \phi(q, u)},$$

where $\varphi(q, u) = o(q, u)$ and $\phi(q, u) = o(q, u)$. The corresponding characteristic equation and its roots have the form

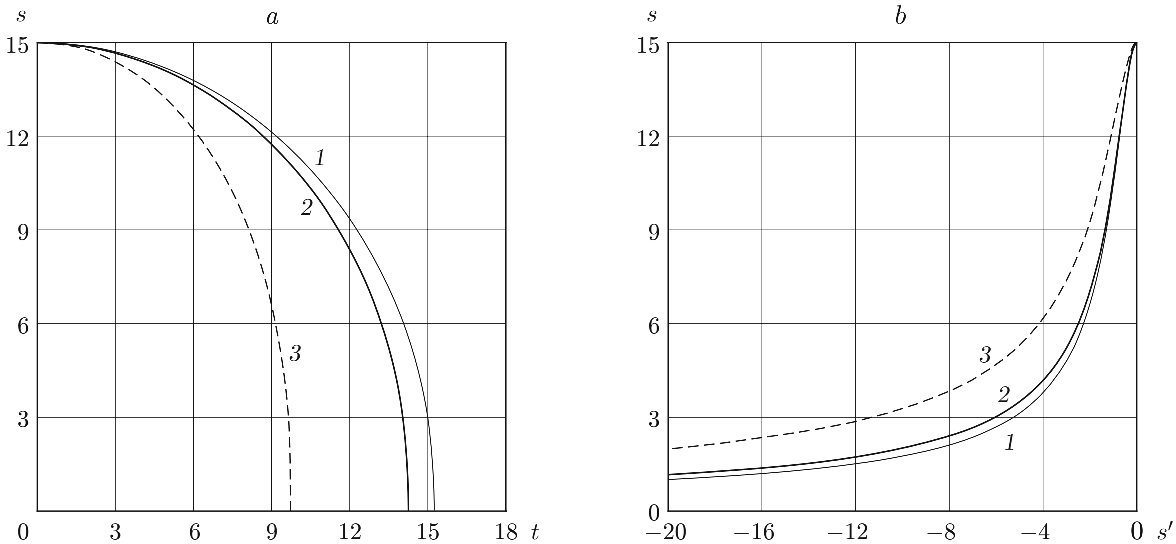

Fig. 2. Cavity radius versus time (a) and velocities of motion of the cavity boundary (b) for Re = 10 and $M = 0.1$: 1) $N = 0$; 2) $N = 1$; 3) $N = 10$.

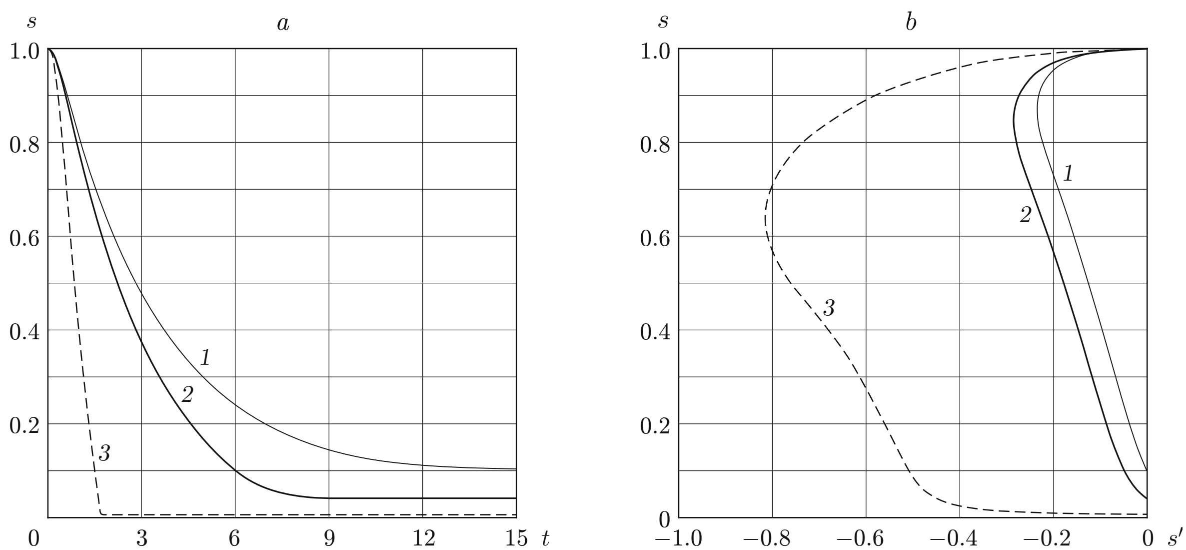

Fig. 3. Cavity radius versus time (a) and velocities of motion of the cavity boundary (b) for Re = 1 and $M = 0.01$: 1) $N = 0$; 2) $N = 0.1$; 3) $N = 1$.

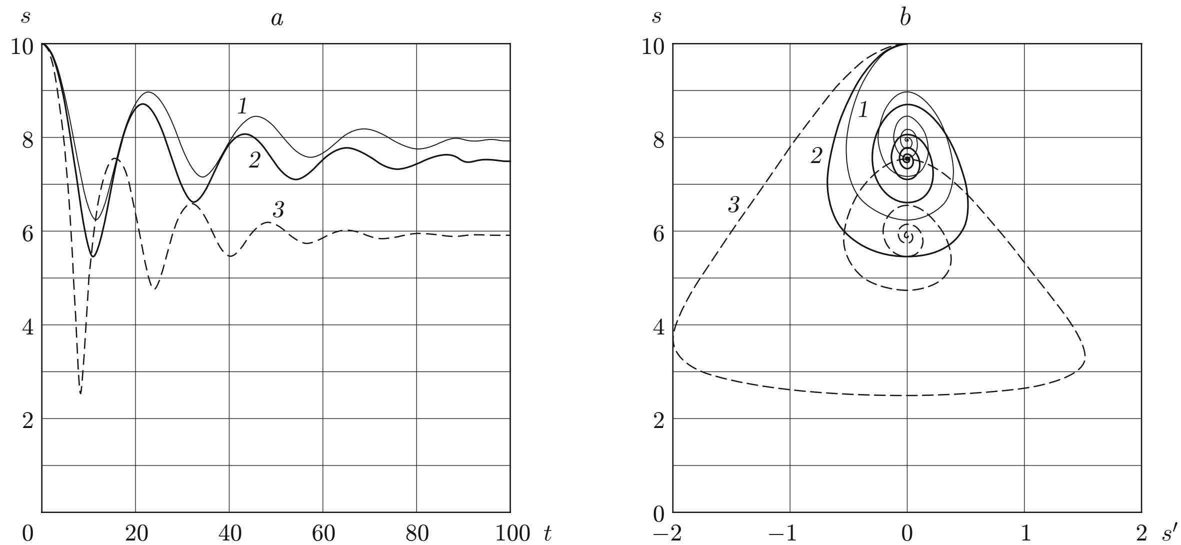

Fig. 4. Cavity radius versus time (a) and velocities of motion of the cavity boundary (b) for Re = 10 and $M = 1$: 1) $N = 0$; 2) $N = 1$; 3) $N = 5$.

$$\lambda^2 + (3\,\mathrm{Re}\,d^2 + 8\,\mathrm{Re}\,d)\lambda + 2\,\mathrm{Re}\,d^3(4\,\mathrm{Re}\,(1-M)d + 12Md^2) = 0,$$

$$\lambda_1 = -\{\mathrm{Re}\,(3d^2+8d) + [\mathrm{Re}^2(3d^2+8d)^2 - 8\,\mathrm{Re}\,d^3(4\,\mathrm{Re}\,(1-M)d + 12Md^2)]^{1/2}\}/2,$$

$$\lambda_2 = -\{\mathrm{Re}\,(3d^2+8d) - [\mathrm{Re}^2(3d^2+8d)^2 - 8\,\mathrm{Re}\,d^3(4\,\mathrm{Re}\,(1-M)d + 12Md^2)]^{1/2}\}/2.$$

It is obvious that if the roots are different and real, they have the same sign. In this case , the singular point is a node, which corresponds to mode 2. We note that the same mode corresponds to the case of a viscous fluid at $M = 0$ and to the case where the roots are equal (in this case, the singular point is a degenerate node). The imaginary roots obviously have a real part; therefore, in this case, the singular point is a focus, which corresponds to mode 3.

To obtain the asymptotic behavior of the velocity when $s(t) \to d$ monotonically, we write $u(s) = A(s-d)^\alpha$. Substitution of this relation into Eq. (17) yields

$$\alpha A^2(s-d)^{2\alpha-1} + \frac{3}{2}\,A^2\,\frac{(s-d)^{2\alpha}}{s} + 4A\,\frac{(s-d)^\alpha}{s^2} + \frac{2M}{\mathrm{Re}}\,\frac{(s-d_1)(s-d_2)(s-d)}{s^3} = 0.$$

We set $\alpha = 1$. Then, from the equation

$$A^2 + 4A/d^2 + 2M(d-d_1)(d-d_2)/(d^3\,\mathrm{Re}) = 0,$$

we determine the quantity A

$$A_{1,2} = \Big[-2d\,\mathrm{Re} \pm \sqrt{4d^2\,\mathrm{Re}^2 - 2M\,\mathrm{Re}\,d^3(d-d_1)(d-d_2)}\,\Big]/(d^3\,\mathrm{Re}),$$

where d_1 and d_2 are the second and third roots of Eq. (16).

As $M \to 0$, the required value of A corresponds to the minus sign. Thus, the velocity tends to zero as a linear function of the cavity boundary:

$$u(s) = \Big[-2d\,\mathrm{Re} - \sqrt{4d^2\,\mathrm{Re}^2 - 2M\,\mathrm{Re}\,d^3(d-d_1)(d-d_2)}\,\Big](s-d)/(d^3\,\mathrm{Re}).$$

2. We examine the behavior of the cavity in a viscoelastic Kelvin–Voigt medium with capillary forces taken into account. In this case, Eq. (12) becomes

$$\ddot{s} + \frac{3}{2}\,\frac{\dot{s}^2}{s} + 4\nu\,\frac{\dot{s}}{s^2} + \Big(\frac{p_0}{\rho} - \varkappa\Big)\frac{1}{s} + \frac{2\sigma}{\rho s^2} + \varkappa\,\frac{2s^3 - a^3}{s^3 a} = 0.$$

In dimensional variables, this equation and the initial conditions are written as

$$\ddot{s} + \frac{3}{2}\,\frac{\dot{s}^2}{s} + 4\,\frac{\dot{s}}{s^2} + (1 - M)\,\frac{1}{s} + 2N\,\frac{1}{s^2} - M\,\mathrm{Re}^2\,\frac{1}{s^3} + 2\,\frac{M}{\mathrm{Re}} = 0; \tag{18}$$

$$s(t) = \mathrm{Re}, \qquad \dot{s}(t) = 0, \qquad t = 0. \tag{19}$$

Here $N = \sigma/(\nu\sqrt{p_0\rho}\,)$.

The solution of problem (18), (19) is sought by numerical integration. Accounting for capillary forces leads to results similar to those considered above. In contrast to the result obtained by Gal'perin and described in [6], in the case of a small initial radius and with capillary forces taken into account, the velocity of motion of the cavity boundary does not tend to a finite nonzero value and the cavity does not collapse in finite time. The mode obtained by Gal'perin with elastic forces taken into account corresponds to mode 2: as the cavity radius tends to a certain positive value, the velocity decreases to zero. The effect of capillary forces is manifested in the fact that the cavity collapses more rapidly than it does only under the action of constant pressure at infinity (see Figs. 2–4). However, once a relatively small radius is reached during the collapse, the elastic forces become dominant and, beginning at a certain time, they prevent the cavity from collapsing, stabilizing its radius at a certain level.

The asymptotic behavior of the cavity boundary in a Kelvin–Voigt medium at the moment of collapse is the same in the presence and absence of capillary forces, which is easily verified by substitution of the relation $s(t) = A(t_* - t)^n$ into Eq. (18).

In the cases where the elastic forces are great and prevent cavity collapse, the limiting value of the boundary c is determined from Eq. (18). The equation for the constant c is written as

$$2Mc^3 + \mathrm{Re}\,(1 + M)c^2 + 2\,\mathrm{Re}\,Nc - M\,\mathrm{Re}^3 = 0. \tag{20}$$

We study Eq. (18) in the phase plane near the singular point $s = c$, $u = 0$. For this, we make the substitution $q = s - c$ and, using relations (16), write (18) as

$$\frac{du}{dq} = -\frac{(4\,\mathrm{Re}\,(N - M + 1)c + 12Mc^2)q + (3\,\mathrm{Re}\,c^2 + 8\,\mathrm{Re}\,c)u + \varphi(q,u)}{2\,\mathrm{Re}\,c^3u + \phi(q,u)},$$

where $\varphi(q,u) = o(q,u)$ and $\phi(q,u) = o(q,u)$. The corresponding characteristic equation and its roots become

$$\lambda^2 + (3\,\mathrm{Re}\,c^2 + 8\,\mathrm{Re}\,c)\lambda + 2\,\mathrm{Re}\,c^3(4\,\mathrm{Re}\,(N - M + 1)c + 12Mc^2) = 0,$$

$$\lambda_1 = -\{\mathrm{Re}\,(3c^2 + 8c) + [\mathrm{Re}^2(3c^2 + 8c)^2 - 8\,\mathrm{Re}\,c^3(4\,\mathrm{Re}\,(N - M + 1)c + 12Mc^2)]^{1/2}\}/2,$$

$$\lambda_2 = -\{\mathrm{Re}(3c^2 + 8c) - [\mathrm{Re}^2(3c^2 + 8c)^2 - 8\,\mathrm{Re}\,c^3(4\,\mathrm{Re}\,(N - M + 1)c + 12Mc^2)]^{1/2}\}/2.$$

The difference between the above relations and the corresponding relations obtained ignoring surface tension forces is that capillary forces add a term to the discriminant of the equation that extends the range corresponding to the singular point — focus.

To construct the asymptotics for the case $s(t) \to c$, we write Eq. (18) in terms of the velocity of motion of the boundary:

$$\frac{du}{ds} + \frac{3}{2}\,\frac{u}{s} + \frac{4}{s^2} + (1 - M)\,\frac{1}{su} + 2\,\frac{N}{s^2u} - M\,\mathrm{Re}^2\,\frac{1}{s^3u} + 2\,\frac{M}{\mathrm{Re}\,u} = 0 \tag{21}$$

and substitute relation $u(s) = A(s - c)^\alpha$ into Eq. (21):

$$\alpha A^2(s - c)^{2\alpha - 1} + \frac{3}{2}\,A^2\,\frac{(s - c)^{2\alpha}}{s} + 4A\,\frac{(s - c)^\alpha}{s^2} + \frac{2M}{\mathrm{Re}}\,\frac{(s - c_1)(s - c_2)(s - c)}{s^3} = 0$$

[c_1 and c_2 are the second and third roots of Eq. (20)]. Similarly to the solution of the problem in the absence of capillary forces, we obtain

$$u(s) = \Big[-2c\,\mathrm{Re} - \sqrt{4c^2\,\mathrm{Re}^2 - 2M\,\mathrm{Re}\,c^3(c - c_1)(c - c_2)}\,\Big](s - c)/(c^3\,\mathrm{Re}).$$

The velocity tends to zero as a linear function of the cavity boundary.

I thank V. V. Pukhnachev for useful comments and great attention to this work and S. V. Stebnovskii for useful discussions and advice.

REFERENCES

1. J. Rayleigh, *The Theory of Sound*, Dover, New York, (1945).
2. H. Poritsky, "The collapse or growth of a spherical bubble or cavity in viscous fluid," in: *Proc. of the 1st United States National Congress of Applied Mechanics* (Chicago, June 17–21, 1951), ASME, New York (1952), pp. 813–821.
3. S. S. Shu,"Note on the collapse of a spherical cavity in a viscous incompressible fluid," *Cal. Inst. Tech. Report.*, **26**, No. 4, 823–825 (1952).
4. A. D. Pernik, *Cavitation Problems* [in Russian] Sudpromgiz, Leningrad (1963).
5. E. I. Zababakhin, "Filling of bubbles in a viscous fluid," *Prikl. Mat. Mekh.*, **24**, No. 6, 1129–1131 (1960).
6. V. K. Andreev, *Stability of the Unsteady Motion of a Fluid with a Free Boundary* [in Russian], Nauka, Novosibirsk (1992).
7. E. Zana and L. G. Leal, "Dissolution of a stationary gas bubble in a quiescent viscoelastic fluid," *Industr. Eng. Chem. Fundam.*, **14**, No. 3, 175–182 (1975).
8. E. Zana and L. G. Leal, "Dynamics and dissolution of gas bubbles in a viscoelastic fluid," *Int. J. Multiphase Flow*, **4**, No. 3, 237–262 (1978).
9. H. J. Yoo and C. D. Han, "Oscillatory behavior of a gas bubble growing (or collapsing) in viscoelastic liquids," *AIChE J.*, **28**, No. 6, 1002–1009 (1982).
10. H. J. Yoo and C. D. Han, "Development of a mathematical model of foam devolatilization," *Polym. Proc. Eng.*, **2**, No. 2, 129–151 (1984).
11. G. Pearson and S. Middleman, "Elongational flow behavior of viscoelastic liquids. Part 1. Modeling of bubble collapse," *AIChE J.*, **23**, No. 5, 714–722 (1977).
12. R. Y. Ting, "Viscoelastic effect of polymers on single bubble dynamics," *AIChE J.*, **21**, No. 4, 810–813 (1975).
13. S. K. Godunov, *Elements of Continuum Mechanics* [in Russian], Nauka, Moscow (1978).
14. P. Germain, *A Course in Mechanics of Continua*, Masson, Paris (1973).